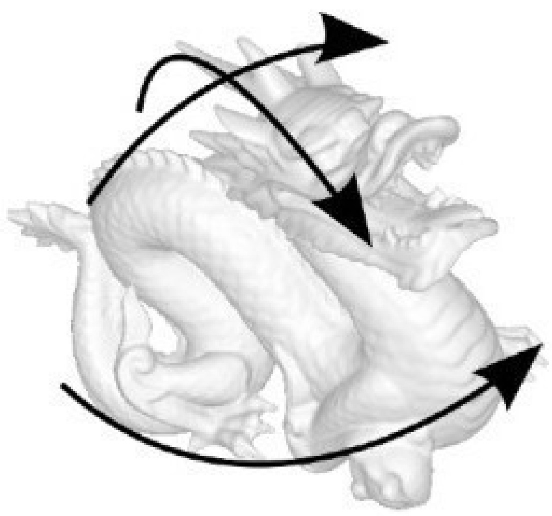
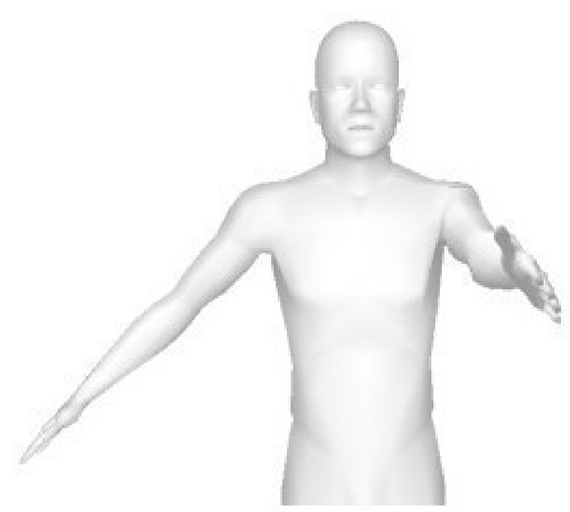




Rotation - Quaternion

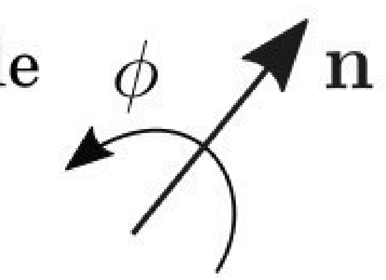
Rotation

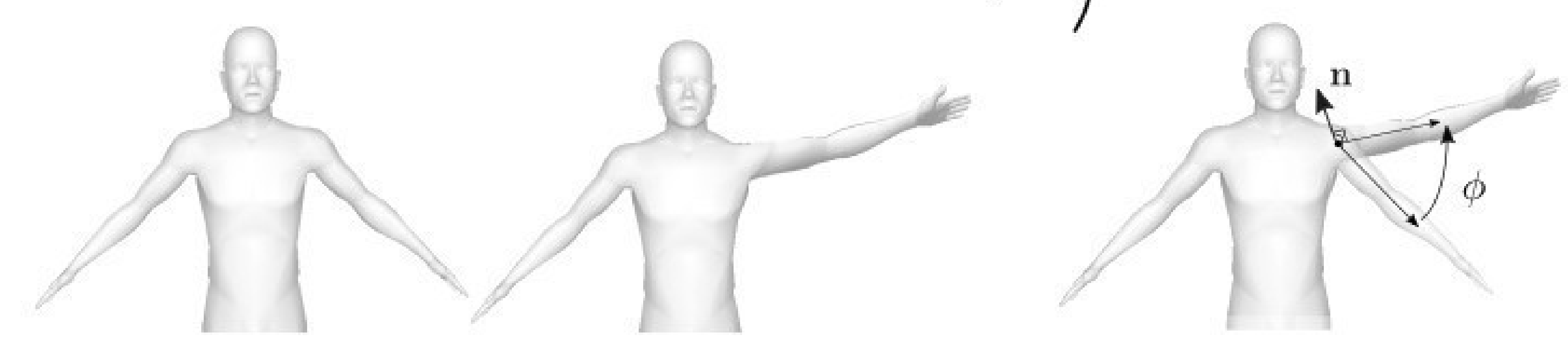
A 3D rotation can be described by:

An orthonormal matrix

$$R(\mathbf{n},\phi) = \begin{pmatrix} \cos(\phi) + n_x^2(1-\cos(\phi)) & n_x n_y(1-\cos(\phi)) - n_z\sin(\phi) & n_y\sin(\phi) + n_x n_z(1-\cos(\phi)) \\ n_z\sin(\phi) + n_x n_y(1-\cos(\phi)) & \cos(\phi) + n_y^2(1-\cos(\phi)) & -n_x\sin(\phi) + n_y n_z(1-\cos(\phi)) \\ -n_y\sin(\phi) + n_x n_z(1-\cos(\phi)) & n_x\sin(\phi) + n_y n_z(1-\cos(\phi)) & \cos(\phi) + n_z n_z(1-\cos(\phi)) \end{pmatrix}$$

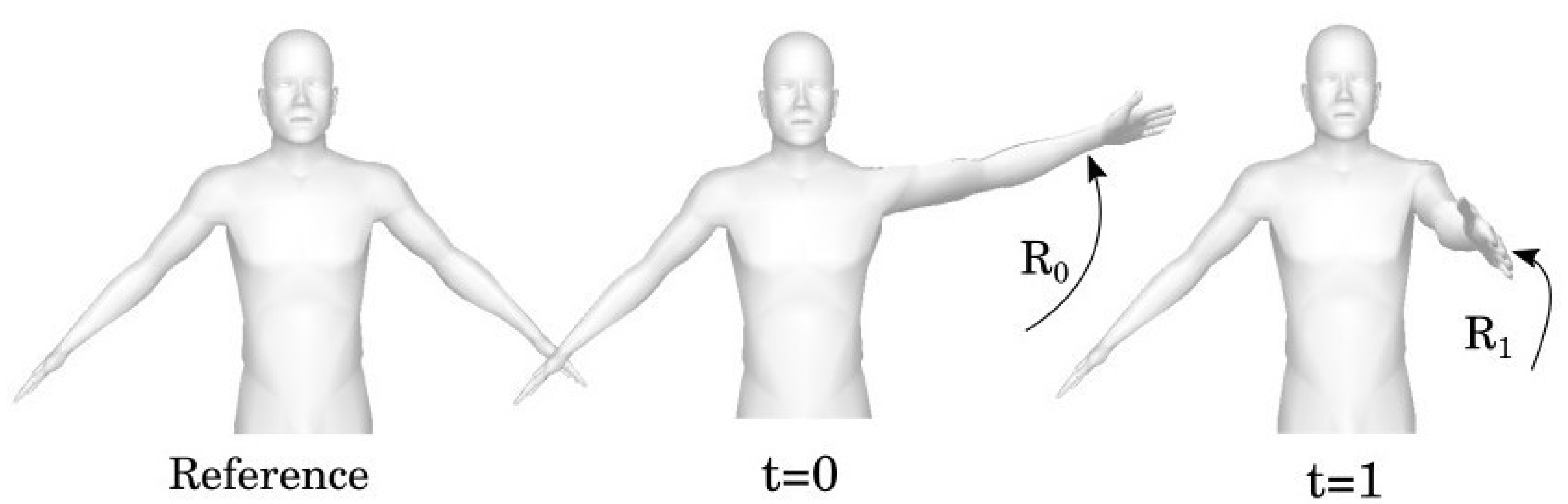
A normalized axe, and an angle





Rotation interpolation

Given 2 rotations, how to interpolate them ?

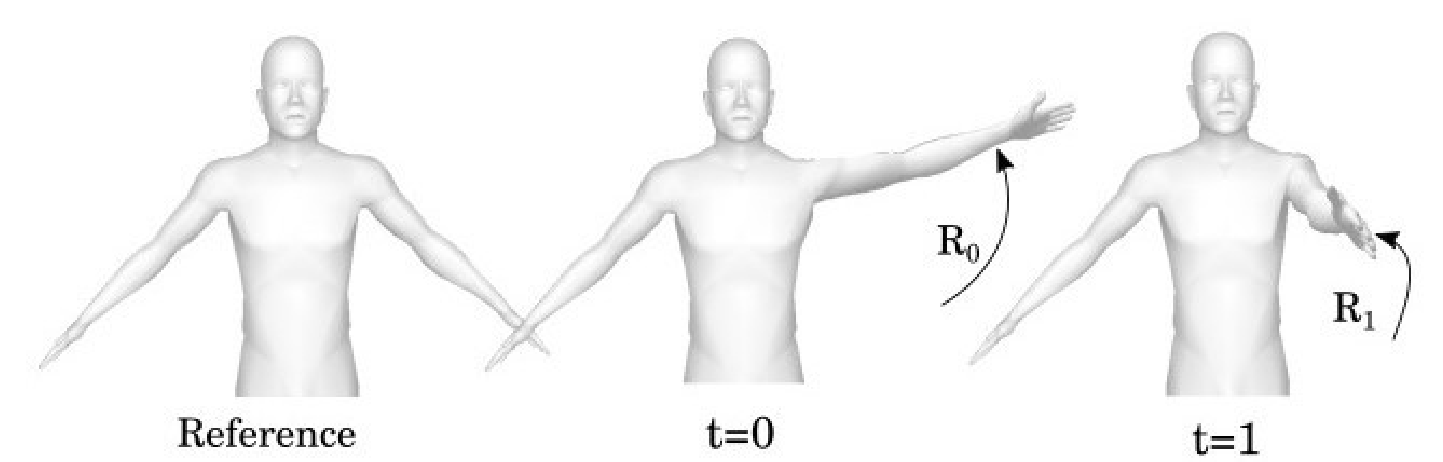


What is $R_{1/2}$ at t=0.5 ?

Problem: 0.5 R_0 + 0.5 R_1 is not a rotation!

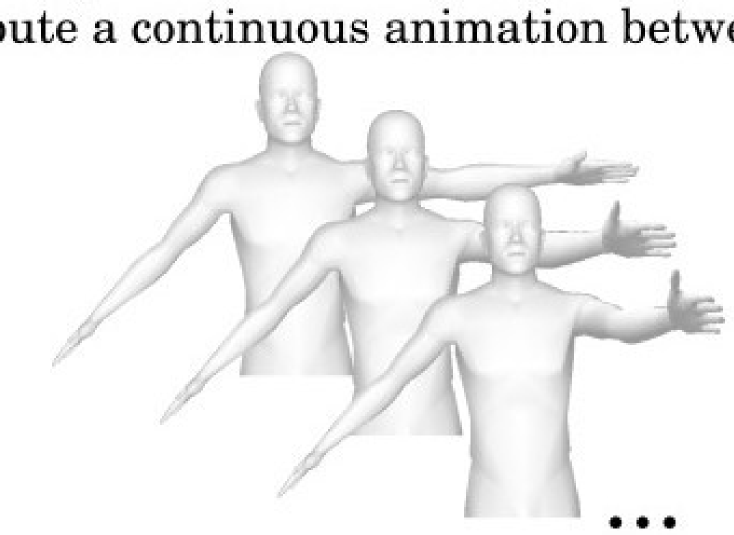
Rotation space is not a vectorial space!

Rotation interpolation



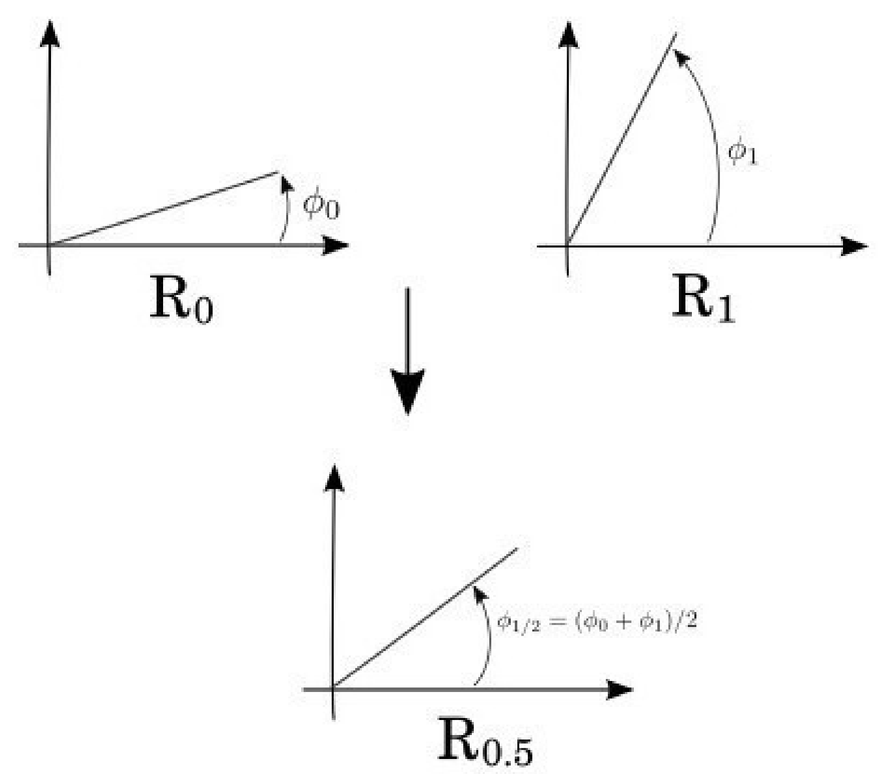
More general question:
Given R_0 and R_1,
How to compute a continuous animation between the keyframe 0 and keyframe 1 ?

R(t=0.1)
R(t=0.2)
R(t=0.3)
...



Rotation interpolation

In 2D, easy solution
=> Interpolate the angle

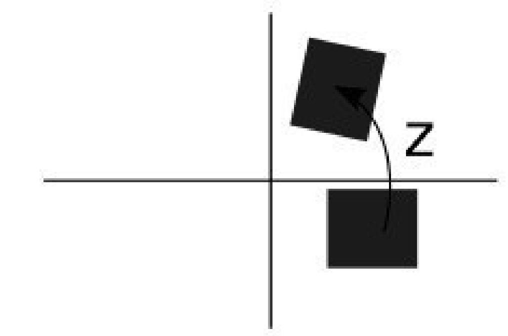


But in 3D, there is also two normalized angles

Complex number

Remember that complex numbers can model rotation in 2D

$$z = e^{\mathbf{i}\theta}$$



$$z = x + y\,\mathbf{i}$$
$$|z| = 1$$

Given 2 rotation

$$z_0 = e^{\mathbf{i}\theta_0}$$
$$z_1 = e^{\mathbf{i}\theta_1}$$

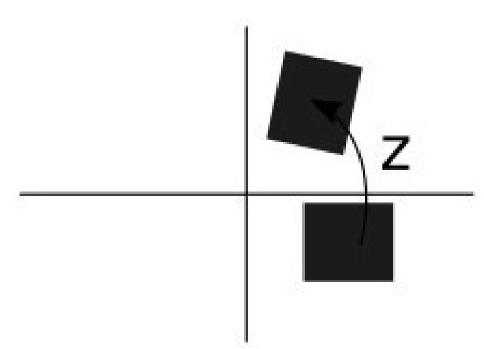
the middle one is

$$z_{1/2} = e^{\mathbf{i}(\theta_0+\theta_1)/2}$$

Complex number

Remember that complex numbers can model rotation in 2D

$$z = e^{\mathbf{i}\theta}$$



$$z = x + y\,\mathbf{i}$$
$$|z| = 1$$

More generally:

Given 2 rotation

$$z_0 = e^{\mathbf{i}\theta_0}$$
$$z_1 = e^{\mathbf{i}\theta_1}$$

the interpolated one is

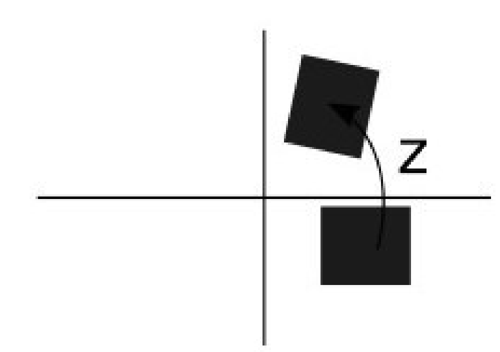
$$z_\alpha = e^{\mathbf{i}((1-\alpha)\theta_0+\alpha\theta_1)}$$
$$\alpha \in [0,1]$$

=> Nice formalism of the complex number for rotation

Generalisation to quaternion

Complex numbers can model rotation in 2D

$$z = e^{\mathbf{i}\theta}$$



$$z = x + y\,\mathbf{i}$$
$$|z| = 1$$

Quaternion can model rotation in 3D

$$q = x + y\,\mathbf{i} + z\,\mathbf{j} + w\,\mathbf{k} \;\in \mathbb{R}^4$$

⚠

$$|q| = 1$$

$$i^2 = -1 \qquad ij = k$$
$$j^2 = -1 \qquad jk = i$$
$$k^2 = -1 \qquad ik = -j \quad ijk = -1$$

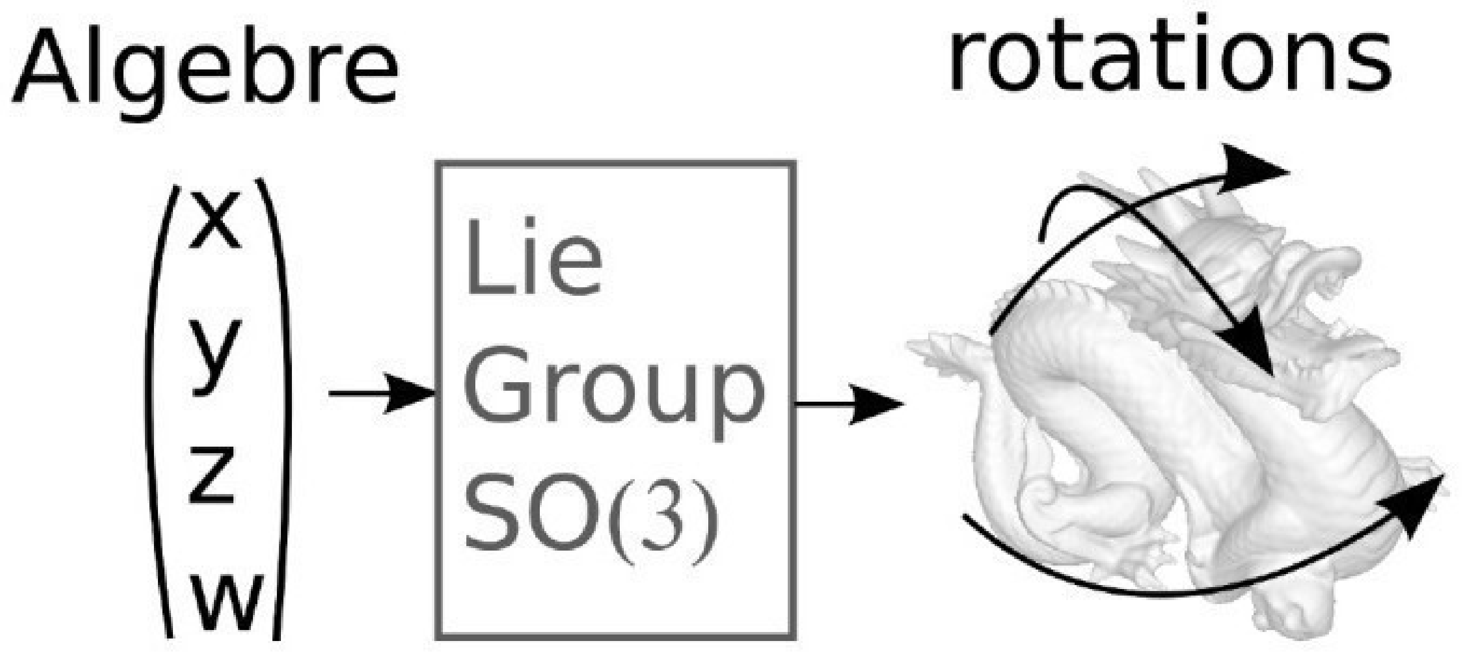
Quaternion

Unit quaternion are rotations in 3D

$$q = x + y\,\mathbf{i} + z\,\mathbf{j} + w\,\mathbf{k}$$
$$|q| = 1$$

+ Short representation of rotation $q_1\, q_2 \Leftrightarrow \mathrm{R}_1\mathrm{R}_2$

(non comutative product)



Equivalence rotation/quaternion

Quaternion $q = (q_0, q_1, q_2, q_3) \;\Rightarrow\;$ Matrix R

$$\mathrm{R} = \begin{pmatrix} 1-2(q_1^2+q_2^2) & 2(q_0q_1 - q_3q_2) & 2(q_0q_2 + q_3q_1) \\ 2(q_0q_1 + q_3q_2) & 1-2(q_0^2+q_2^2) & 2(q_1q_2 - q_3q_0) \\ 2(q_0q_2 - q_3q_1) & 2(q_1q_2 + q_3q_0) & 1-2(q_0^2+q_1^2) \end{pmatrix}$$

Efficient representation of multiplication: $q = (\mathbf{v}, w)$

$$q_0\; q_1 = (w_0\mathbf{v}_1 + w_1\mathbf{v}_0 - \mathbf{v}_0 \times \mathbf{v}_1, w_0\,w_1 - \mathbf{v}_0 \cdot \mathbf{v}_1)$$

Applying a rotation to a 3D point

Given a 3D point $p = (x, y, z)$

And a unit quaternion q

The rotation of p by q is given by p'

where $p' = (z'_0, z'_1, z'_2)$

with $\boxed{z' = q\; z\; \overline{q}}$

$\overline{q} = (-q_0, -q_1, -q_2, q_3)$ is the conjugated quaternion.

Quaternion interpolation

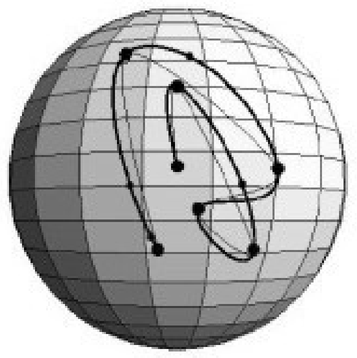
Quaternion/rotation can be interpolated using the SLERP.

Spherical Linear intERPoltation

$$\mathrm{SLERP}(q_0, q_1, \alpha) = \frac{\sin(\alpha(1-\theta))}{\sin(\theta)} q_0 + \frac{\sin(\alpha\theta)}{\sin(\theta)} q_1$$

$\cos(\theta) = q_0 \cdot q_1$

$\alpha \in [0, 1]$



Other approach

Matrix interpolation also exists.

Generalization of linear interpolation in Lie Group

$$R = \left(R_2\, R_1^{-1}\right)^{\alpha} R_1$$

But: requires matrix exponential

- Computational cost
- Numerical instabilities

=> Quaternion SLERP is more efficient

Application to camera trajectory