Spline, N.U.R.B.S, etc.

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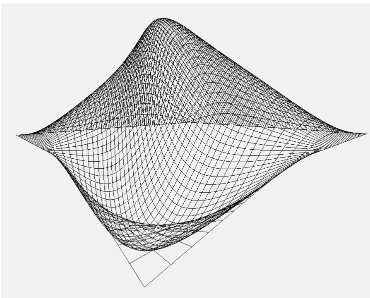
Notes

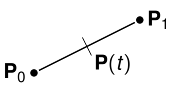
Objectif



- Surface lisse
- Controlable localement
- Forme attendue (prévisible)

Notes

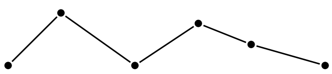
Segments



$$\begin{cases} \mathbf{P}(0) = \mathbf{P}_0 \\ \mathbf{P}(1) = \mathbf{P}_1 \end{cases}$$

$$\mathbf{P}(t) = (1-t) \cdot \mathbf{P}_0 + t \cdot \mathbf{P}_1$$

Extension : Courbe brisée

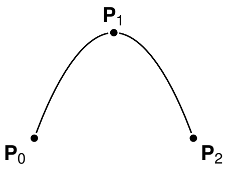


Problème : courbe non lisse

Notes

Parabole

degré 2 $\Rightarrow$ 3 points



$$\begin{cases} \mathbf{P}(0) = \mathbf{P}_0 \\ \mathbf{P}(\frac{1}{2}) = \mathbf{P}_1 \\ \mathbf{P}(1) = \mathbf{P}_2 \end{cases}$$

Notes

Parabole résolution

$$\mathbf{P}(t) = a \cdot t^2 + b \cdot t + c$$

$$\begin{cases} c & = & \mathbf{P}_0 \\ \frac{a}{4} + \frac{b}{2} + c & = & \mathbf{P}_1 \\ a + b + c & = & \mathbf{P}_2 \end{cases}$$

$$\begin{cases} c & = & \mathbf{P}_0 \\ a + 2 \cdot b & = & 4 \cdot (\mathbf{P}_1 - \mathbf{P}_0) \\ a + b & = & \mathbf{P}_2 - \mathbf{P}_0 \end{cases}$$

$$b = 4 \cdot \mathbf{P}_1 - 4 \cdot \mathbf{P}_0 - \mathbf{P}_2 + \mathbf{P}_0 = 4 \cdot \mathbf{P}_1 - 3 \cdot \mathbf{P}_0 - \mathbf{P}_2$$

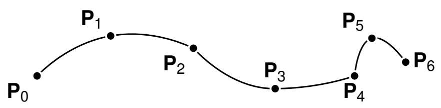
$$a = \mathbf{P}_2 - \mathbf{P}_0 - 4 \cdot \mathbf{P}_1 + 3 \cdot \mathbf{P}_0 + \mathbf{P}_2 = 2 \cdot \mathbf{P}_2 + 2 \cdot \mathbf{P}_0 - 4 \cdot \mathbf{P}_1$$

$$\left[\mathbf{P}(t) = 2 \cdot (\mathbf{P}_0 - 2 \cdot \mathbf{P}_1 + \mathbf{P}_2) \cdot t^2 + (-3 \cdot \mathbf{P}_0 + 4 \cdot \mathbf{P}_1 - \mathbf{P}_2) \cdot t + \mathbf{P}_0\right]$$

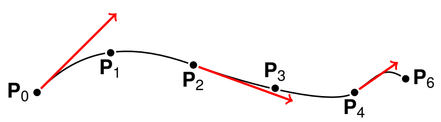
Notes

Courbes raccordables

Problème



On aimerait



2 positions + 2 dérivées (directions) $\Rightarrow$ polynôme de degré 3

Notes

Cubique d'Hermite

$$\begin{cases} \mathbf{P}(0) &= \mathbf{P}_0 \\ \mathbf{P}(1) &= \mathbf{P}_1 \\ \mathbf{P}'(0) &= d_0 \\ \mathbf{P}'(1) &= d_1 \end{cases}$$

$$\mathbf{P}(t) = a \cdot t^3 + b \cdot t^2 + c \cdot t + d$$

$$\mathbf{P}'(t) = 3 \cdot a \cdot t^2 + 2 \cdot b \cdot t + c$$

soit

$$\begin{cases} a + b + c + d = \mathbf{P}_1 \\ 3 \cdot a + 2 \cdot b + c = d_1 \\ c = d_0 \\ d = \mathbf{P}_0 \end{cases}$$

Notes

Courbes raccordables, démonstration

$$\begin{cases} a + b = \mathbf{P}_1 - \mathbf{P}_0 - d_0 \\ 3 \cdot a + 2 \cdot b = - d_0 + d_1 \end{cases}$$

$$a = -2 \cdot \mathbf{P}_1 + 2 \cdot \mathbf{P}_0 + d_0 + d_1 = 2 \cdot (\mathbf{P}_0 - \mathbf{P}_1) + d_1 + d_0$$

$$b = \mathbf{P}_1 - \mathbf{P}_0 - d_0 + 2 \cdot \mathbf{P}_1 - 2 \cdot \mathbf{P}_0 - d_0 - d_1 = 3 \cdot (\mathbf{P}_1 - \mathbf{P}_0) - 2 \cdot d_0 - d_1$$

$$\begin{bmatrix} \mathbf{P}(t) = [2 \cdot (\mathbf{P}_0 - \mathbf{P}_1) + d_0 + d_1] \cdot t^3 + \\ [3 \cdot (\mathbf{P}_1 - \mathbf{P}_0) - 2 \cdot d_0 - d_1] \cdot t^2 \quad + d_0 \cdot t + \mathbf{P}_0 \end{bmatrix}$$

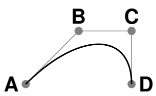
Note : Manipuler des dérivées, c'est pratique pour des courbes, mais nettement moins pour des surfaces !

Notes

Représentation de Bézier

$$\mathbf{P}(t) = (2 \cdot t^3 - 3 \cdot t^2 + 1) \cdot \mathbf{P}_0 + (-2 \cdot t^3 + 3 \cdot t^2) \cdot \mathbf{P}_1 + (t^3 - 2 \cdot t^2 + t) \cdot d_0 + (t^3 - t^2) \cdot d_1$$

$$\mathbf{P}(t) = (1-t)^3 \underbrace{\mathbf{P}_0}_{\mathrm{A}} + 3 \cdot (1-t)^2 \cdot t \cdot \underbrace{(\mathbf{P}_0 + \frac{d_0}{3})}_{\mathrm{B}} + 3 \cdot (1-t) \cdot t^2 \cdot \underbrace{(\mathbf{P}_1 - \frac{d_1}{3})}_{\mathrm{C}} + t^3 \cdot \underbrace{\mathbf{P}_1}_{\mathrm{D}}$$

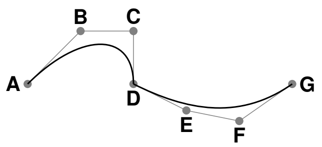


- Polygone de controle **A B C D**
- Courbe tangente à [**A B**] et à [**C D**]
- Courbe qui passe par **A** et **D**

Notes

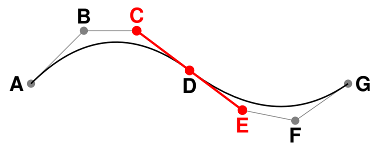
Raccordement de courbes de Bézier

Problème :
C^0 uniquement
par défaut



Solution

Pour du C^1 : **E** symétrique de **C** par rapport à **D**

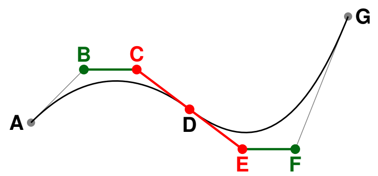


Notes

Raccordement C^2 de courbes de Bézier

Pour du C^2 :

- **E** symétrique de **C** par rapport à **D**
- **F** symétrique de **B** par rapport à **D**



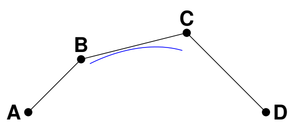
On est donc assez restreint sur les positions des points.

Notes

B-Spline, définition

$$
\begin{aligned}
\mathbf{P}(t) = & \tfrac{1}{6}(1 - 3t + 3t^2 - t^3)\cdot \mathbf{A} \\
+ & \tfrac{1}{6}(4 - 6t^2 + 3t^3)\cdot \mathbf{B} \\
+ & \tfrac{1}{6}(1 + 3t + 3t^2 - 3t^3)\cdot \mathbf{C} \\
+ & \tfrac{1}{6} t^3 \cdot \mathbf{D}
\end{aligned}
$$

$$\mathbf{P}(t) = \sum_{k=0}^{n} b_k(t)\cdot \mathbf{P}_k$$



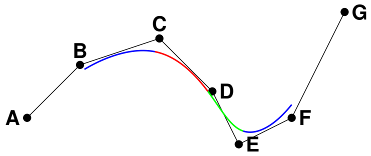
Note : La courbe ne commence pas par **A** ni ne finit par **D**.

Notes

B-Spline, avantages

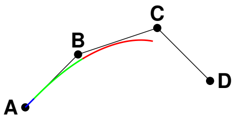
- Automatiquement raccordable C^2

ABCD
BCDE
CDEF
DEFG



- Peut passer par des points (en les multipliants)

ABCD
AABC
AAAB



Notes

Expression matricielle

Bézier :

$$\mathbf{P}(t) = (1-t)^3 \cdot \mathbf{P}_0 + 3 \cdot (1-t)^2 \cdot t\mathbf{P}_1 + 3 \cdot (1-t) \cdot t^2\mathbf{P}_2 + t^3\mathbf{P}_3$$

$$\begin{array}{rcccccl}
= & (1 & -3 \cdot t & +3 \cdot t^2 & -t^3) & \mathbf{P}_0 \\
+3\cdot & (& t & -2 \cdot t^2 & +t^3) & \mathbf{P}_1 \\
+3\cdot & (& & t^2 & -t^3) & \mathbf{P}_2 \\
+ & & & & t^3 & \mathbf{P}_3
\end{array}$$

$$\begin{array}{rl}
= (-\mathbf{P}_0 + 3 \cdot \mathbf{P}_1 - 3 \cdot \mathbf{P}_2 + \mathbf{P}_3 &) \cdot t^3 + \\
3 \cdot (\mathbf{P}_0 - 2 \cdot \mathbf{P}_1 + \mathbf{P}_2 &) \cdot t^2 + \\
3 \cdot (-\mathbf{P}_0 + \mathbf{P}_1 &) \cdot t + \\
\mathbf{P}_0 &
\end{array}$$

$$= \begin{pmatrix} \mathbf{P}_0 & \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} t^3 \\ t^2 \\ t \\ 1 \end{pmatrix}$$

Notes

Expression matricielle, suite

$$= \mathbf{P}^{\mathrm{T}} \cdot M \cdot T$$

$$= T^{\mathrm{T}} \cdot M \cdot \mathbf{P} = \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix} \cdot (M) \cdot \begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix}$$

B-Spline
On remplace simplement M par :

$$M_s = \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Notes

Passage de 1D vers 2D

Produit tensoriel :

- 1D :

$$f(t) = \sum_k b_k(t)\mathbf{P}_k$$

- 2D :

$$f(u,v) = \sum_{k_u}\sum_{k_v} b_{k_u}(u) b_{k_v}(v)\mathbf{P}_{k_u k_v}$$

Notes

Pour une surface de Bézier

$$f(u,v) = \sum_{k_u=0}^{3}\sum_{k_v=0}^{3} b_{k_u}(u) b_{k_v}(v)\mathbf{P}_{k_u k_v}$$

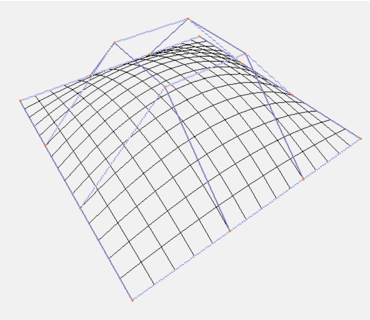
$$f(u,v) = \sum_{k_u=0}^{3} b_{k_u}(u) \underbrace{\left(\sum_{k_v=0}^{3} b_{k_v}(v)\mathbf{P}_{k_u k_v}\right)}_{[\mathbf{P}_{k_u O} \ldots \mathbf{P}_{k_u 3}] \cdot M \cdot V}$$

$$f(u,v) = U^T \cdot M \cdot [\mathbf{P}] \cdot M \cdot V$$

$$f(u,v) = \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \cdot M \begin{pmatrix} \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} & \mathbf{P}_{03} \\ \mathbf{P}_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} \\ \mathbf{P}_{20} & \mathbf{P}_{21} & \mathbf{P}_{22} & \mathbf{P}_{23} \\ \mathbf{P}_{30} & \mathbf{P}_{31} & \mathbf{P}_{32} & \mathbf{P}_{33} \end{pmatrix} \cdot M \begin{pmatrix} v^3 \\ v^2 \\ v \\ 1 \end{pmatrix}$$

Notes

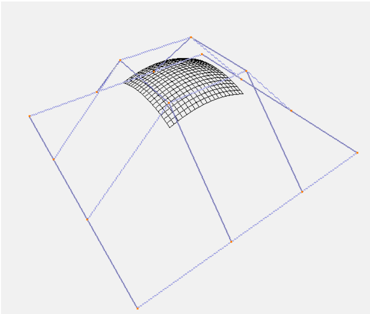
Surface de Bézier, représentation graphique



Notes

Surface Spline, raccordable

Comme pour les surfaces de Bézier, seule la matrice M change.



Notes

Bibliographie

- *The Nurbs Book* Les Piegl & Wayne Tiller
- « Cours écrit » de Damien Rohmer

Notes

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